ON THE THEORY OF RAYLEIGH TYPE WAVES IN AN ANISOTROPIC HALF-SPACE

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A solution representing Rayleigh type waves in the half-space under consideration was constructed for the plane case in [1] by using the method of complex solutions of Smirnov-Sobolev [2]; the question of existence of Rayleigh type waves is investigated.

The study of existence of Rayleigh type waves for an anisotropic half-space reduces, in contrast to the isotropic case, to an investigation of a Rayleigh type function and some algebraic functions entering the solution. This question was considered in [1], while additions to this paper are made below.

The solutions (71), (72) and (77_1) (77_2) in [1] which express Rayleigh type waves in an anisotropic half-space can be represented by a single form

$$\begin{aligned} u_1 &= R\{c\theta\lambda_1 F_2(\theta) w_1(t+\theta x+\lambda_1 y)\} \\ v_1 &= R\{(a\theta^2+d\lambda_1^2-1)\ F_2(\theta)\ w_1(t+\theta x+\lambda_1 y)\} \\ u_2 &= R\{-c\theta\lambda_2 F_1(\theta) w_1(t+\theta x+\lambda_2 y)\} \\ v_2 &= R\{-(a\theta^2+d\lambda_2^2-1)F_1(\theta) w(t+\theta x+\lambda_2 y)\} \\ F_k &= a\theta^2-(c-d)\lambda_k^2-1 \qquad (k=1,2) \end{aligned} \tag{1}$$

for values of the complex variable θ equal to the roots of a Rayleigh type equation $\theta = \pm 1/R$.

The values λ_1 and λ_2 are here

$$\lambda_k = \{A(\theta)+(-1)^k[A^2(\theta)-B(\theta)]^{1/2}\}^{1/2} \qquad (k=1,2) \tag{2}$$

$$A(\theta) = \frac{(b+d)-(ab+d^2-c^2)\theta^2}{2bd}, \qquad B(\theta) = \frac{a}{b}\left(\frac{1}{a}-\theta^2\right)\left(\frac{1}{d}-\theta^2\right)$$

The elastic constants or real anisotropic media of the class under consideration satisfy the conditions

$$a > d, \qquad b > d, \qquad d > 0, \qquad ab-(c-d)^2 > 0 \tag{3}$$

which express the necessary and sufficient conditions for positive definiteness of the elastic energy form.

It has been established in [1] that for any values of the elastic constants which satisfy conditions (3), the Rayleigh type function for the half-space under consideration has two real roots $\theta = \pm 1/R$ which are within the intervals $(\pm 1/\sqrt{d}, \pm\infty)$. Therefore, the solution (1) exists for any value of the elastic constants.

There remains to ascertain that the solution (1) is, for any value of the elastic constants satisfying the conditions (3), a Rayleigh type wave whose action is manifested principally on the half-space boundary $y = 0$ and damps rapidly as it recedes therefrom. To do this it is sufficient to show that for $\theta = \pm 1/R$ the algebraic functions (2) have imaginary or complex values.

An attempt was made to solve this question in [1] by assuming that the following three cases are possible for the quantities in (2)

$$A(\pm 1/R) < 0, \quad A^2(\pm 1/R) - B(\pm 1/R) > 0 \tag{4}$$

$$A^2(\pm 1/R) - B(\pm 1 / R) < 0 \quad (5)$$

$$A(\pm 1 / R) > 0, \quad A^2(\pm 1 / R) - B(\pm 1 / R) > 0 \quad (6)$$

The first two cases result in the assertion that the solutions (1) are Rayleigh type waves; the third case, associated with satisfaction of condition (6), was not studied in [1].

If conditions (6) actually hold for some elastic constants, then for $\theta = \pm 1 / R$ the functions (2) will have real values. Hence, Rayleigh type waves do not exist in this case, the solution (1) will express some other kind of waves.

The question was not studied in [1] whether all three cases expressed by conditions (4) – (6) can hold, and if so then under which conditions does any case hold for the elastic constants. Therefore, the question has not been solved whether Rayleigh type waves can exist for all media of the class of anisotropy under consideration, and if not for all, then for which conditions can such waves exist for the elastic constants.

Let us examine these questions by utilizing the results of investigating the algebraic functions (2) obtained by the author in [3, 4]. It must be kept in mind that in [3]

$$n_k = l\lambda_k, \quad \vartheta = l / m = 1 / \theta$$

Upon compliance with conditions (3) the following inequalities can hold

$$(a - d)b - c^2 > 0 \quad (7)$$

$$(a - d)b - c^2 < 0 \quad (8)$$

Upon compliance with condition (7), the functions (2) have imaginary or complex values on the $(\pm 1 / \sqrt{d}, \pm \infty)$ sections of the real axis of the complex θ-plane, depending on compliance with some additional conditions on the elastic constants indicated in [3].

Therefore, the functions (2) take on imaginary or complex values for values of the roots of a Rayleigh type equation $\theta = \pm 1/R$, and the solution (1) expresses a Rayleigh type wave.

If condition (8) is satisfied, the functions (2) have real values on the sections $(\pm 1 / \sqrt{d}, \pm \theta_1^0)$ and complex ones on the sections $(\pm \theta_1^\circ, \pm \infty)$. The points $\pm \theta_1^\circ$ are here real zeros of the functions within the inner radicand in (2) and are determined by the expression [4]

$$\theta_1^\circ = \left(\frac{M - \sqrt{4bdc^2[c^2 - (a-d)(b-d)]}}{K_1K_2}\right)^{1/2} \quad (9)$$

$$K_1 = ab - (c - d)^2, \qquad K_2 = ab - (c + d)^2$$

$$M = (b + d)[(a - d)(b - d) - c^2] - (a - b)(b - d)d$$

Let us establish to which of these sections the roots of a Rayleigh type equation can belong for (8).

Function of Rayleigh type on sections $(\pm 1 / \sqrt{d} \pm \infty)$ of the real axis of the first sheet of a Riemann surface has the form [1]

$$R(\theta) = \mp iR_1(\theta)$$

$$R_1(\theta) = \{[ab - (c - d)^2]\,\theta^2 - b\}\sqrt{\theta^2 - 1/d} - \sqrt{ab}\sqrt{\theta^2 - 1/a} \quad (10)$$

where

$$R_1(\pm 1 / \sqrt{d}) < 0, \qquad R_1(\pm \infty) > 0 \quad (11)$$

on the boundaries of these sections.

Substituting (9) into (10), we obtain

$$R_1(\pm\theta_1^\circ) = \frac{\sqrt{dc^2}\,[D^+ - \sqrt{bN}]\,[\sqrt{bN} - D^-]}{\sqrt{d}K_2\sqrt{M + \sqrt{4bdc^2N}}}, \quad D^\pm = \sqrt{d}(b-d) \pm \sqrt{dc^2} \tag{12}$$

Upon compliance with (8), we have [3]

$$K_2 < 0, \qquad N = c^2 - (a-d)\ (b-d) > 0 \tag{13}$$

The differences between the squares in the square brackets in (12) are

$$\begin{aligned} &[\sqrt{d}(b-d) + \sqrt{dc^2}]^2 - [\sqrt{bN}]^2 = (b-d)K_1 > 0 \\ &[\sqrt{bN}]^2 - [\sqrt{d}(b-d) - \sqrt{dc^2}]^2 = -(b-d)K_2 > 0 \end{aligned} \tag{14}$$

For quantities under the outer radical in the denominator of (12) we have

$$\sqrt{4bdc^2N} - |M| > 0 \tag{15}$$

since for (8)

$$4bdc^2N - M^2 = -(b-d)^2K_1K_2 > 0$$

It follows from (12) – (15) that $R_1(\theta_1^\circ) < 0$. Then, according to (11) we can assert that under the condition (8) the roots of an equation of Rayleigh type are within the sections $(\pm\theta_1^\circ,\ \pm\infty)$.

Therefore, in the case of compliance with condition (8), the functions (2) have complex values for values of the roots of a Rayleigh type equation $\theta = \pm 1/R$, and the solution (1) is a Rayleigh type wave.

Let us note, in conclusion, that the investigations performed permitted a complete study of the question, examined in [1], of the existence of a Rayleigh type wave on the free boundary of an anisotropic half-space with four elastic constants.

The possibility has been established of the existence of Rayleigh type waves for any real values of the elastic constants for media of the anisotropy class under consideration.

A study of the Rayleigh type functions was continued It has been established that under the condition (7) the roots of a Rayleigh type equation are within the sections $(\pm 1/\sqrt{d},\ \pm\infty)$, and under the condition (8) within the sections $(\pm\theta_1^\circ,\ \pm\infty)$. Therefore, the velocity of Rayleigh type waves R does not exceed the value $\sqrt{d}$ under the condition (7), and the value $1/\theta_1^0$ under condition (8).

It has been shown that for any values of the elstic constants the functions (2) take on imaginary or complex values for values of the roots of a Rayleigh type equation. Therefore, of the three cases proposed in [1], cases (4) and (5) can hold depending on the elastic constants, while the third case (6) cannot hold.

BIBLIOGRAPHY

1. Sveklo, V. A., Elastic oscillations of an anisotropic solid. Uch. Zap., Leningrad Univ., Ser. Matem. Nauk, №17, 1949.
2. Frank, F. and Mises, R., Differential and Integral Equations of Mathematical Physics, Part 2. Moscow-Leningrad, ONTI, 1937.
3. Osipov, I. O., On a method of a functional-invariant solutions for problems of dynamic elasticity theory of anisotropic media. Izv. Akad. Nauk SSSR, Ser. geofiz., №3, 1963.
4. Osipov, I. O., On the two-dimensional problem of propagation of elastic waves due to a point source in an anisotropic medium. PMM Vol 33, №3, 1969.

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